

Stochastic Resonance: influence of a $f^{-\kappa}$ noise spectrum

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Abstract

Here, in order to study *stochastic resonance* (SR) in a double-well potential when the noise source has a spectral density of the form $f^{-\kappa}$ with varying κ , we have extended a procedure, introduced by Kaulakys et al (Phys. Rev. E **70**, 020101 (2004)). In order to have an analytical understanding of the results, we have obtained an effective Markovian approximation, that allows us to make a systematic study of the effect of such kind of noises on the SR phenomenon. The comparison of numerical and analytical results shows an excellent qualitative agreement indicating that the effective Markovian approximation is able to correctly describe the general trends.

PACS numbers:

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I. INTRODUCTION

Stochastic resonance (SR) is one of the most interesting *noise-induced phenomena*, that arises from the interplay between *deterministic* and *random* dynamics in a *nonlinear* system [1]. This phenomenon has been largely studied during more than two decades due to its great interest not only from a basic point of view but also for its technological interest and biological implications [1, 2].

Most of those studies have used white or colored noises, with a few exceptions where more wide classes of noises were considered. For instance, in [3] SR in systems subject to a colored and non Gaussian noise was studied. However, there are other works more tightly related with the present one, as indicated by the following examples. In [4], the authors studied, through numerical simulations, the behavior of the signal-to-noise ratio (SNR) gain in a level crossing detector and a Schmidt trigger, when subject to a colored noise composed of a periodic train pulse plus a Gaussian $f^{-\kappa}$ noise with variable κ . Their results indicate that the maximum of the SNR is larger for white noise, and moves towards large noise intensities for increasing κ . In [5] experimental evidence was found that noise can enhance the homeostatic function in the human blood pressure regulatory system. Related with the last work, other experimental evidence was found in [6] that an externally applied f^{-1} noise, added to the usual white noise, contributes to sensitizing the baroflex function in the human brain. In [7], and in a model of traffic junction of a main and a side road, it was found that the effect of a Gaussian $f^{-\kappa}$ noise with $\kappa \geq 0$ shows an overall traffic efficiency enhancement. An enhancement of the SR phenomenon in a FitzHugh-Nagumo model submitted to a colored noise with $f^{-\kappa}$ for $0 \leq \kappa \leq 2$ was found in [8]. In [9] it was experimentally demonstrated that an SR-like effect can be obtained in rat sensory neurones with white, f^{-1} and f^{-2} noises, and that, under some particular conditions, f^{-1} noise can be better than white noise to enhance neuron's response. Related to it, in [10] it was shown that it is possible to enhance the SR effect in a FitzHugh-Nagumo model submitted to a colored noise with $f^{-\kappa}$, and that the optimal noise variance of SR could be minimized with $\kappa \approx 1$.

Motivated by the work of Kaulakys and collaborators [11], who have introduced a method to generate f^{-1} noises over a wide range of frequencies (see also [12, 13]), here we discuss how to extend such a procedure for positive and negative values of the stochastic variable.

We also exploit this procedure to analyze the effect of a noise spectrum of the form $f^{-\kappa}$ with varying κ , on the SR phenomenon in a simple double-well potential. In the following Section we present the model system to be studied and the procedure to generate the $f^{-\kappa}$ noise. Afterwards we discuss an effective Markovian approximation, and exploit it to study SR. Finally we discuss the results and draw some general conclusions.

II. THE SYSTEM

A. Stochastic differential equations

The starting point of our analysis is the following system of stochastic differential equations

$$\dot{x} = f(x) + g(x)y(t) \quad (1)$$

$$\dot{y} = \frac{u(y)}{\tau} + \frac{D}{\tau}v(y)\xi(t), \quad (2)$$

where x is the coordinate of a particle diffusing in a double well potential $U_o(x) = -\int^x f(\zeta)d\zeta = \frac{x^4}{4} - \frac{x^2}{2}$, subject to a noise $y(t)$. The second equation corresponds to the Langevin equation driving the noise $y(t)$, inspired in the work of Kaulakis et al. [11]. In this last equation we consider a new potential $V(y) = -\int^y u(s)ds$, and a (white) noise $\xi(t)$ that enters in a multiplicative form with a function $v(y)$. The last function will be characterized by an exponent μ .

We consider the following form for the function $u(y)$

$$u(y) = \alpha y^3 - \beta y^5 + s(y)y^4 \quad (3)$$

where $s(y)$ indicates the sign of y (i.e, -1 if $y < 0$ and $+1$ if $y \geq 0$). For the function $v(y)$ we adopt

$$v(y) = |y|^\mu + c, \quad (4)$$

where both, the exponent μ and the constant c are positive (> 0).

The above indicated forms change the symmetry of the potential $V(y)$ and, in addition, when compared with the work in [11], increases the range of the noise variable from $[0, +\infty)$ to $(-\infty, +\infty)$, as is shown in Fig. 1. The parameter c allows for the random variable y to adopt negative values when $c > |y|^\mu$.

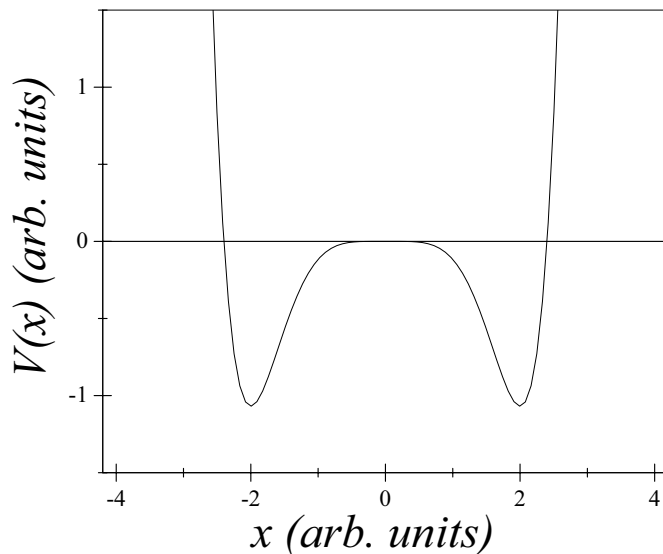


FIG. 1: Symmetric potential $V(y)$ as derived from Eq. (3). We have used the following values: $\alpha = 5 \times 10^{-4}$ and $\beta = \frac{1}{2}$.

B. Characteristic of the noise variable y

The most relevant aspect of the process y is its power spectral density (PSD) with a $1/f$ frequency behavior. Kaulakis et al [11] have shown that when $c = 0$ and $\mu = 5/2$, the noise y exhibits a $1/f$ functionality in a wide range of frequencies. For $c > 0$, but small, this property is still valid, as we show in figures 2 and 3.

When the exponent μ changes, the PSD behaves as $1/f^k$, with $k < 1$ for $\mu < 5/2$. We will use this property to evaluate the mean-first-passage-time (MFPT), and the signal-to-noise ratio (SNR). In particular we have used $\mu = 3/2$, yielding $k \simeq 3/4$.

III. EFFECTIVE MARKOVIAN THEORY

In order to be able to obtain some analytical results, we resort to an *effective Markovian approximation*, analogous to the so called *unified colored noise approximation* (UCNA) [14, 15]. Exploiting this approach we are able to find an effective Markovian Fokker-Planck equation (FPE) for the probability density $P(x, t)$. The procedure is the following (the prime will indicate derivation respect to the variable x).

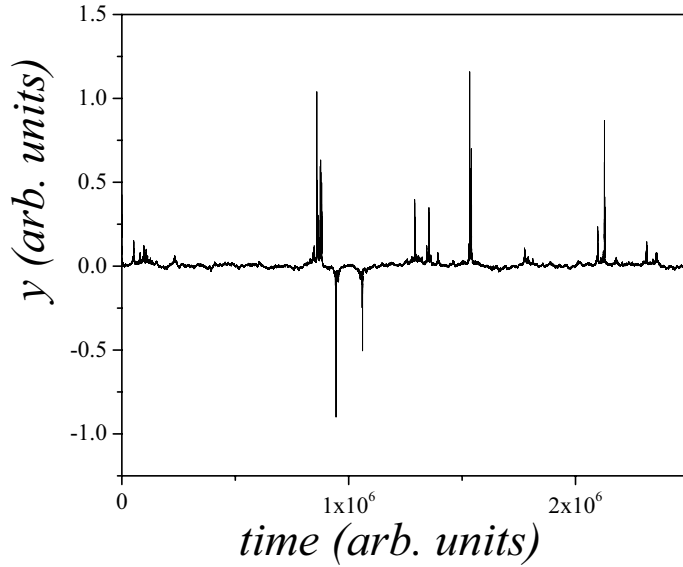


FIG. 2: A typical realization for Eq. (2) using $\mu=5/2$ and $c = 1 \times 10^{-4}$. The PSD of this realization shows a $1/f$ behavior, see Fig. 3.

A. Adiabatic procedure

Deriving Eq. (1) respect to the time we have

$$\ddot{x} = f'(x)\dot{x} + g'(x)\dot{x}y + g(x)\dot{y}. \quad (5)$$

Now, assuming an adiabatic behavior, we eliminate \ddot{x} , and using Eq. (2) we obtain

$$0 \simeq f'(x)\dot{x} + g'(x)\dot{x} \left[\frac{\dot{x} - f(x)}{g(x)} \right] + g(x) \left[\frac{u(Z(x))}{\tau} + \frac{Dv(Z(x))\xi(t)}{\tau} \right], \quad (6)$$

where we have defined $Z(x) = Z_0(x) + Z_1(x)$, with $Z_0(x) = -\frac{f(x)}{g(x)}$ and $Z_1(x) = \frac{\dot{x}}{g(x)}$. Now, we use the following approximation

$$u(Z(x)) \approx u(Z_0(x)) + u'(Z_0(x)) Z_1(x) \quad (7)$$

and similarly for v

$$v(Z) \approx v(Z_0) + v'(Z_0) Z_1(x). \quad (8)$$

Adopting now $g(x) = 1$, that implies $Z_0 \equiv -f(x)$, we have

$$0 = f'(x)\dot{x} + \frac{u(Z_0(x))}{\tau}\dot{x} + D\frac{v(Z_0(x))}{\tau}\xi(t) + O(\dot{x}\xi(t)). \quad (9)$$

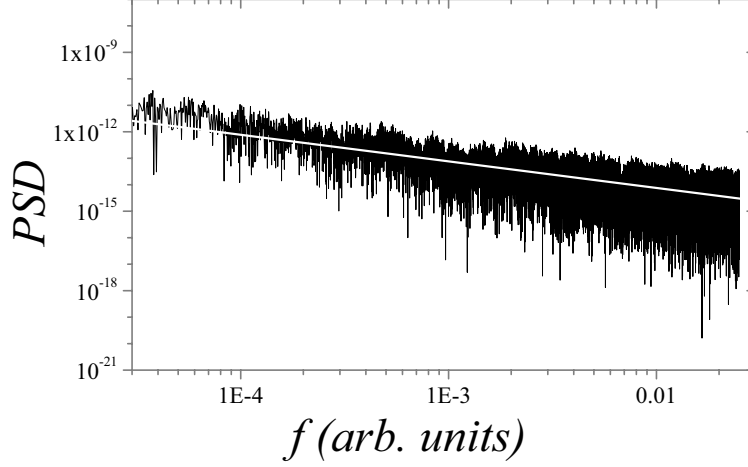


FIG. 3: PSD for the variable y , as indicated in Eq. (2). We used the same values of parameters as in Fig. 2. The white line corresponds to a linear fitting, resulting in a slope $\kappa = -1.004 \pm 0.005$.

With the above indicated results, the effective equation for the process x adopts the following form

$$\dot{x} = -\frac{u(Z_0(x)) + Dv(Z_0(x))\xi(t)}{\tau f'(x) + u'(Z_0(x))} = A_1(x) + B_1(x)\xi(t), \quad (10)$$

and, due to the polynomial character of the function $h(y)$, we can write the following limit for $A(x)$ and $B(x)$ when $\tau \rightarrow 0$

$$A_1(x) \rightarrow \frac{Z_0(x)}{3} = A(x), \quad (11)$$

$$B_1(x) \rightarrow -\frac{Dv(Z_0(x))}{u'(Z_0(x))} = B(x). \quad (12)$$

Finally, using the above indicated approximations, the stochastic differential equation for the process x reads

$$\dot{x} = A(x) + B(x)\xi(t). \quad (13)$$

B. Fokker-Planck equation

The FPE associated with the Langevin equation, Eq. (13), is (using the Ito prescription [16])

$$\frac{\partial}{\partial t}P(x, t) = -\frac{\partial}{\partial x}[A(x)P(x, t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B^2(x)P(x, t)]. \quad (14)$$

As is well known, the stationary distribution of this FPE is given by

$$P(x) = \frac{N}{B(x)} \exp\{-\Phi(x)\}, \quad (15)$$

where N is the normalization factor, and

$$\Phi(x) = 2 \int^x \frac{A(\zeta)}{B(\zeta)} d\zeta. \quad (16)$$

C. Mean-First-Passage-Time and SR

The indicated FPE and its associated stationary distribution allow us to obtain the mean-first-passage-time (MFPT) through a Kramers-like approximation. Using known expressions we obtain for the MFPT [16]

$$T(x_0) = 2 \int_a^{x_0} \frac{dy}{\Psi(y)} \int_{-\infty}^y \frac{dz \Psi(z)}{B(z)}, \quad (17)$$

where

$$\Psi(x) = \exp\left\{2 \int^x \frac{A(\zeta)}{B(\zeta)} d\zeta\right\}. \quad (18)$$

In order to study SR, as usual we introduce an external signal in the form of a term rocking the double well potential: $U(x) = U_o(x) + S(t)$, with $S(t) = S_o \sin(\omega t)$ (in what follows we adopt $\omega = 1.33 \times 10^{-5}$). Exploiting the so called “two-state approximation” [1], we define the SNR as the ratio of the strength of the output signal and the broadband noise output evaluated at the signal frequency ω , obtaining [1]

$$SNR \propto \left\{ \frac{1}{T} \frac{dT}{dS} \right\}_{S=0}, \quad (19)$$

where the derivative of the T in the above expression, as indicated, is evaluated at $S = 0$.

IV. RESULTS AND CONCLUSION

We have done extensive numerical simulations of the full set of Eqs. (1,2) in order to obtain the SNR. The results are shown in Fig. 4. Also, in Fig. 5 we show the SNR computed using the effective Markovian theory, obtained through Eqs. (17) and (19). In order to be able to compare the results we have normalized the curves. Also, in order to have a well defined variance of the noise process, in all the simulation we have obtained numerically the variance of $y(t)$, as described by the Eq. (2).

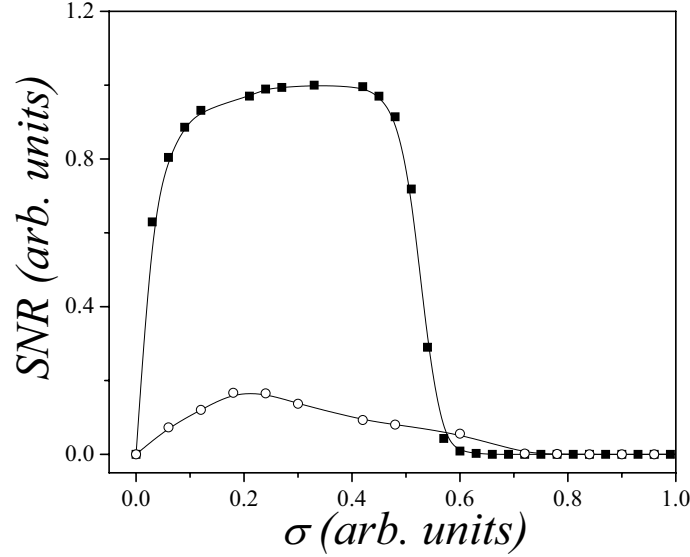


FIG. 4: SNR obtained when simulating the full set of Eq. (1,2). Here σ corresponds to the noise intensity defined through the distribution width, as indicated in the text. Squares and circles corresponds for $\mu = 3/2$ and $\mu = 5/2$ respectively. The lines are for guiding the eye only.

From the comparison of both figures it is apparent that the results obtained using the effective Markovian theory are in very good (qualitative) agreement with those from simulations. This is in accord with previous results obtained for different systems [3, 15]. We can conclude that such kind of UCNA-like approximation offers an adequate framework to obtain effective Markovian approximations in a very wider class of systems than the one to which was originally applied [14].

The above indicated results are in complete agreement with those of [4]. That is: the maximum of the SNR is larger for white noise, and it moves towards large noise intensities

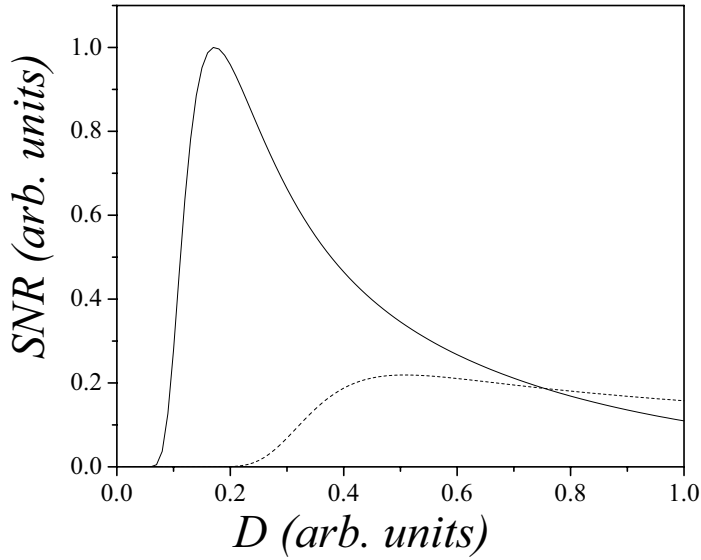


FIG. 5: SNR obtained using Eq. (19), as derived from the two state theory. Continue and dashed line correspond to $\mu = 3/2$ and $\mu = 5/2$ respectively.

for increasing κ . In order to gain some physical insight about this behavior it is worth to remark that the function defined by Eq. (18) is directly related to an effective potential within the approximation we used

$$V_{eff}(x) \approx D \ln \Psi(x) = D \left\{ 2 \int^x \frac{A(\zeta)}{B(\zeta)} d\zeta \right\}. \quad (20)$$

The behavior of such a potential reveals what are the consequences of changing the exponent μ : when $\mu = 5/2$ (i.e. the PSD is $1/f$), the effective potential shows a well defined well; but as μ decreases the well is less defined as shown in Fig. 6. Such a behavior of the effective potential explains why the SNR increases when μ decreases. The general theory shows that the SNR increase is proportional to the Kramers rate r_K , that is given by

$$r_K = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{\Delta V_{eff}}{D} \right\}, \quad (21)$$

where ΔV_{eff} is the high of the barrier in the effective potential separating the attractors. Hence, the reduction of the SNR with increasing κ (or decreasing μ) could be directly related with the marked reduction of the barrier separating the attractors in the effective potential picture. This also explains the reason for the shift of the SNR maximum towards larger values of the noise intensity.

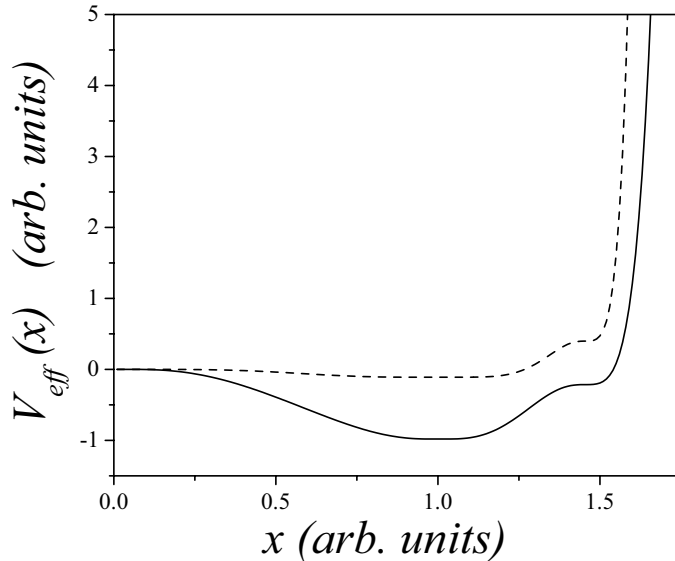


FIG. 6: Effective potential, Eq (18), showing the different behavior of the wells for different values of the exponent μ : continuous line $\mu = 5/2$, dotted line $\mu = 3/2$. Due to the symmetry, we only show positive values of x .

In conclusion, we want to remark that this is a first step towards an analytical understanding of two very important, connected, and ubiquitous, aspects in natural processes. Those are: the $1/f^k$ behavior of noises' PSD, and its role in signal detection via the SR mechanism. The physical picture provided by the indicated effective Markovian approximation, offers an adequate framework to analyze and understand the main qualitative trends of such phenomenon. Even more, we expect to apply the same scheme to other noise induced phenomena when subject to $1/f^k$ noises. This will be the subject of further work.

Acknowledgments: MAF would like to thank the support of CONICET, Argentina, to the Santa Fe Institute for its support and hospitality, and to M. Ballard for fruitful discussions related to the one-dimension FPE description. HSW wants to thank the European Commission for the award of a *Marie Curie Chair*.

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